MATH 320 NOTES, WEEKS 12 AND 13

So, far we know that if $A \in M_{n,n}(F)$, the following are equivalent:

(1) A is invertible.
(2) $rank(A) = n$.
(3) $Ax = \mathbf{0}$ has only the trivial solution.
(4) $Ax = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.

Next we will define **the determinant** of a matrix, $\det(A) \in F$, and show that the above hold iff $\det(A) \neq 0$. So, computing the determinant will be one more way of deciding if A is invertible.

Section 4.1 Determinants of order 2

Definition 1. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *The determinant of* A *is* $\det(A) = ad - bc$.

Some remarks:

(1) $\det(I_2) = 1$, the determinant of the zero matrix is 0.
(2) If A has a zero row or a zero column, $\det(A) = 0$.
(3) If the row of A are multiples of each other, then $\det(A) = 0$. That's because if $a = kc, b = kd$, we have $ad - bc = kcd - kdc = 0$.

It turns out that the converse of the last item is also true:

Theorem 2. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Then* A *is invertible iff* $\det(A) \neq 0$.

Proof. The easy direction: If A is not invertible, then $rank(A) < 2$, so the rows are linearly dependent, so they are multiples of each other, and so by the above note, $\det(A) = 0$.

Now for the harder direction: If A is invertible, then $rank(A) = 2$, and so A cannot have a zero row. So, $a \neq 0$ or $b \neq 0$ (or both). Suppose $a \neq 0$ (the other case is similar). Then by the type 3 elementary row operation, $R_2 - \frac{c}{a}R_1$, we obtain the matrix

$$B = \begin{pmatrix} a & b \\ 0 & d - \frac{cb}{a} \end{pmatrix}$$

Since elementary row operations preserve the rank, we have that $rank(B) = rank(A) = 2$, and so B is invertible. Then B cannot have a row of zeros. It follows that $d - \frac{cb}{a} \neq 0$, and so $da \neq cb$. Then $\det(A) = ad - cb \neq 0$. $\square$

Lemma 3. *Suppose* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is invertible. Then* $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Proof. Calculate $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = I_2$ $\square$

As a function $\det : M_{2,2}(F) \to F$ is *not* a linear transformation, but it is something close.

Lemma 4. *The function* $\det : M_{2,2}(F) \to F$ *is a linear function of each row, when the other one is fixed:*

- $\det \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c & d \end{pmatrix} = \det \begin{pmatrix} a_1 & b_1 \\ c & d \end{pmatrix} + \det \begin{pmatrix} a_2 & b_2 \\ c & d \end{pmatrix}$.
- $\det \begin{pmatrix} ka & kb \\ c & d \end{pmatrix} = k \ \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Similarly, if we fix the first row.

Section 4.2 Determinants of order n.

First let us introduce some notation. Let $A \in M_{n\times n}(F)$. We will denote the (i,j)-th entry by a_{ij}. Also, given $1 \leq i, j \leq n$, let $\overline{A}_{ij} \in M_{n-1\times n-1}(F)$ be the submatrix obtained by removing the i-th row and the j-th column of A.

Definition 5. *Let* $A \in M_{n\times n}(F)$. *If* $n = 1$, $\det(A) = A = a_{11}$. *If* $n > 1$, *then*

$$\det(A) = \Sigma_{k=1}^{n} (-1)^{k+1} a_{1k} \cdot \det(\overline{A}_{1k}).$$

Note that this definition is by induction on n. I.e. we assume we know the definition of determinant of dimension $(n-1)\times(n-1)$, and use it to define the determinant in the case of dimension $n \times n$. Also, the above formula is computing the determinant of A along *the first row.* Later we will see that we can compute it along any row or column.

Exercise: Verify that for $n = 2$, the above formula gives the same definition as in the last section.

The next lemma is the generalization of Lemma 4 for n by n matrices.

Lemma 6. *The function* $\det : M_{n\times n}(F) \to F$ *is a linear function of each row, when the other ones are fixed. More precisely,*

$$\det \begin{pmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{r-1}} \\ \mathbf{u} + d\mathbf{v} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{pmatrix} = \det \begin{pmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{r-1}} \\ \mathbf{u} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{pmatrix} + d \det \begin{pmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{r-1}} \\ \mathbf{v} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{pmatrix}.$$

Proof. The proof is by induction on n. If $n = 1$, it is clear, so suppose $n > 1$.

Let $A = \begin{pmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{r-1}} \\ \mathbf{u} + d\mathbf{v} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{pmatrix}$; $B = \begin{pmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{r-1}} \\ \mathbf{u} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{pmatrix}$; $C = \begin{pmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{r-1}} \\ \mathbf{v} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{pmatrix}$

We want to show that $\det(A) = \det(B) + d\det(C)$. There are two cases.

Case 1. $r = 1$. Then, by our definition, $\det(A) =$

$$\Sigma_{k=1}^n (-1)^{k+1} a_{1k} \cdot \det(\overline{A}_{1k}) = \Sigma_{k=1}^n (-1)^{k+1} (b_{1k} + dc_{1k}) \cdot \det(\overline{A}_{1k}) =$$

$$\Sigma_{k=1}^n (-1)^{k+1} b_{1k} \cdot \det(\overline{A}_{1k}) + d\Sigma_{k=1}^n (-1)^{k+1} c_{1k} \cdot \det(\overline{A}_{1k}) = \det(B) + d\det(C)$$

Note that here for every k, $\overline{A}_{1k} = \overline{C}_{1k} = \overline{B}_{1k}$, because the only difference in the matrices A, B, and C is the the first row.

Case 2. $r > 1$. Then $\det(A) = \Sigma_{k=1}^n (-1)^{k+1} a_{1k} \cdot \det(\overline{A}_{1k})$ and by the inductive hypothesis, for each k,

$$\det(\overline{A}_{1k}) = \det(\overline{B}_{1k}) + d\det(\overline{C}_{1k}).$$

This is because, the submatrices have dimension $(n-1) \times (n-1)$, and the $(r-1)$-th row of $\overline{A}_{1k}$ equals the $(r-1)$-th row of $\overline{B}_{1k}$ plus d times the $(r-1)$-th row of $\overline{C}_{1k}$. Plugging in, we have,

$\det(A) =$
$= \Sigma_{k=1}^n (-1)^{k+1} a_{1k} \cdot (\det(\overline{B}_{1k}) + d\det(\overline{C}_{1k})) =$
$= \Sigma_{k=1}^n (-1)^{k+1} a_{1k} \cdot \det(\overline{B}_{1k}) + d\Sigma_{k=1}^n (-1)^{k+1} a_{1k} \cdot \det(\overline{C}_{1k}) =$
$= \det(B) + d\det(C)$. □

Corollary 7. *If A has a row of zeros, then* $\det(A) = 0$.

Proof. Say the r-th row of A has only zeros, i.e. this row is $\mathbf{0} = 0 \cdot \mathbf{0}$. Then by the above lemma applied to row r, we have that $\det(A) = 0 \cdot \det(A) = 0$. □

Our next goal is to show that we can compute the determinant by expanding along any row. First we show it in the simplest case – when the row in question is of the form e_k, for some k, i.e. the vector with 1 in the k-th coordinate and 0s everywhere else.

Lemma 8. *Let $A \in M_{n\times n}(F)$ and suppose that the r-th row of A is e_k. Here $1 \leq k, r \leq n$ and $1 < n$. Then* $\det(A) = (-1)^{r+k}\det(\overline{A}_{rk})$.

Proof. By induction on n. For $n = 2$, it is an exercise to verify it.

Suppose $n > 2$, and we have the result for smaller dimensions. Again we divide it into two cases.

Case 1. $k = 1$. Then $a_{1k} = 1$ and for all $j \neq k$, $a_{1j} = 0$. So by the formula for the determinant, we have $\det(A) = (-1)^{1+k} \det(\overline{A}_{1k}) = (-1)^{r+k} \det(\overline{A}_{rk})$.

Case 2. $k > 1$. Then by the inductive hypothesis for each submatrix $\overline{A}_{1j}$,

- if $j = k$, the $r - 1$-th row is $\mathbf{0}$ (because we have removed the k-th column on A). Then $\det(\overline{A}_{1k}) = 0$ by the above corollary.
- if $j \neq k$, the $r - 1$-th row is e_k (with one less dimension). Then, by induction,
  (1) $\det(\overline{A}_{1j}) = (-1)^{r-1+k-1} \det(C_{rj})$, if $j < k$
  (2) $\det(\overline{A}_{1j}) = (-1)^{r-1+k} \det(C_{rj})$, if $j > k$
  where C_{rj} is the submatrix of $\overline{A}_{1j}$ by removing the r-th row and j-th column of $\overline{A}_{1j}$.

When we plug this information if the formula for the determinant, after some computation, we get the desired result. For details, see pg 214 in the textbook. □

Now we can finally prove a very useful fact about determinants: the we can compute them by expanding along *any* row of A.

Theorem 9. *Let $A \in M_{n\times n}(F)$, and let $1 \leq i \leq n$. We can compute* $\det(A)$ *by expanding along row i as follows:* $\det(A) = \Sigma_{j=1}^{n} (-1)^{i+j} a_{ik} \cdot \det(\overline{A}_{ij})$.

Proof. Denote the i-th row of A by $a_i = \langle a_{i1}, a_{i2}, ..., a_{in}\rangle = a_{i1}e_1 + ... + a_{in}e_n$. Fir each $j \leq n$, let $B_j \in M_{n\times n}(F)$ be the matrix obtained by replacing a_i with e_j, i.e. A and B_j differ only in row i. (For example the ith row of B_1 is $\langle 1, 0, ..., 0\rangle$ and every other row is like in A.)

By Lemma 8, we have that for each j, $\det(B_j) = (-1)^{i+j} \det(\overline{(B_j)}_{ij})$. Since $\overline{(B_j)}_{ij}$ is obtained from B_j by removing the i-th row and the j-column. Since the only difference between A and each B_j is in the i-th row, it follows that $\overline{(B_j)}_{ij} = \overline{A}_{ij}$. Plugging in, we get

$$\det(B_j) = (-1)^{i+j} \det(\overline{A}_{ij}).$$

Then, by linearity (Lemma 6), we have that

$$\det(A) = \Sigma_{j=1}^{n} a_{ij} \det(B_j) = \Sigma_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\overline{A}_{ij}).$$

□

Using the above theorem, next we will show what effect doing elementary row operations have on the determinant of a matrix.

Lemma 10. *(Elementary row operations and the determinant) Suppose $A \in M_{n\times n}(F)$ and B is obtained from A by doing one elementary row operation.*

(1) *(Type 1) If B is obtained from A by interchanging two rows, then* $\det(B) = -\det(A)$.

(2) *(Type 2) If B is obtained from A by multiplying one row by k, then* $\det(B) = k\det(A)$.
(3) *(Type 3) If B is obtained from A by adding a multiple of one row to another, then* $\det(B) = \det(A)$.

Proof. **Part 1**. By induction on n. If $n = 2$, it is a straightforward calculation. Suppose that we interchange rows r and k. Let $i < n$, $i \neq r, i \neq k$. Expanding along row i, we get, $\det(B) = \Sigma_{j=1}^{n}(-1)^{i+j}b_{ij}\det(\overline{B}_{ij})$.

Now, for each j, $\overline{B}_{ij}$, is obtained from $\overline{A}_{ij}$ by interchanging rows r and k, and so by induction, $\det(\overline{B}_{ij}) = -\det(\overline{A}_{ij})$. Also, since the i-th row of A and B are the same, we have that $a_{ij} = b_{ij}$ for all $j \leq n$. So,

$$\det(B) = \Sigma_{j=1}^{n}(-1)^{i+j}b_{ij}\det(\overline{B}_{ij}) = \Sigma_{j=1}^{n}(-1)^{i+j}a_{ij}(-\det(\overline{A}_{ij})) =$$
$$-\Sigma_{j=1}^{n}(-1)^{i+j}a_{ij}\det(\overline{A}_{ij}) = -\det(A).$$

Part 2. Suppose that B is obtained by multiplying row i by k. Expanding along row i, we have that

$$\det(B) = \Sigma_{j=1}^{n}(-1)^{i+j}b_{ij}\det(\overline{B}_{ij}) =$$

$$\Sigma_{j=1}^{n}(-1)^{i+j}ka_{ij}\det(\overline{A}_{ij}) = k\Sigma_{j=1}^{n}(-1)^{i+j}a_{ij}\det(\overline{A}_{ij}) = k\det(A).$$

Here $\overline{A}_{ij} = \overline{B}_{ij}$, because we only changed row i.

Part 3. Again, this is by induction on n. Suppose B is obtained from A by adding a multiple of row r to row k. If $n = 2$, this can be verified directly. Otherwise, let $i \leq n, i \neq r, i \neq k$. Then for all $j \leq n$, $\overline{B}_{ij}$ is obtained from $\overline{A}_{ij}$ by adding the same multiple of of row r to row k. So, by induction, $\det(\overline{B}_{ij}) = \det(\overline{A}_{ij})$. Expanding along row i, we have that

$$\det(B) = \Sigma_{j=1}^{n}(-1)^{i+j}b_{ij}\det(\overline{B}_{ij}) = \Sigma_{j=1}^{n}(-1)^{i+j}a_{ij}\det(\overline{A}_{ij}) = \det(A).$$

□

The following is an immediate corollary:

Corollary 11. *If $A, B \in M_{n\times n}(F)$ are such that B is obtained from A by doing elementary row operations, then* $\det(B) = 0$ *iff* $\det(A) = 0$.

Theorem 12. *Let $A \in M_{n\times n}$. Then* $\det(A) = 0$ *iff $rank(A) < n$ iff A is not invertible.*

Proof. We already know that $rank(A) < n$ iff A is not invertible from previous sections. So, we just have to show this is equivalent to $\det(A) = 0$.

For one direction, suppose $rank(A) < n$. Then its rows must be linearly dependent, so there is some row, say row i, which can be written as a linear combination of the other rows. Then by doing type 3 elementary row operations, adding multiples of other rows to row i, we obtain a matrix B from A, such that the i-th row of B is all zeros. Then $\det(B) = 0$. But since doing type 3 elementary row operations don't change the determinant, we also have $\det(A) = 0$.

For the other direction, suppose that $rank(A) = n$. Then by doing elementary row operations, we can obtain I_n from A. Since $\det(I_n) = 1 \neq 0$, then $\det(A) \neq 0$.

$\square$

Section 4.3 A couple of more properties of the determinant.

We start with two lemmas that follow from the effect of doing elementary row operations on the determinant.

Lemma 13. *Suppose that E is an elementary row matrix.*

(1) *(Type 1) If E is obtained by interchanging two rows of I_n, then* $\det(E) = -1$*;*
(2) *(Type 2) If E is obtained from I_n by multiplying a row by k, then* $\det(E) = k$*;*
(3) *(Type 3) If E is obtained from I_n by adding a multiple of one row to another, then* $\det(E) = 1$*;*

Proof. The proof is immediate using Lemma 9 and that $\det(I_n) = 1$. $\square$

Lemma 14. *Suppose that $A = EB$, where E is an elementary (row) matrix, then* $\det(A) = \det(E) \cdot \det(B)$

Proof.
(1) (Type 1) If E is obtained by interchanging two rows of I_n, then $\det(E) = -1$ and A is obtained from B by interchanging the same two rows. So

$$\det(A) = -\det(B) = \det(E) \cdot \det(B);$$

(2) (Type 2) If E is obtained from I_n by multiplying a row by k, $\det(E) = k$ and A is obtained from B by multiplying the same row by k. So

$$\det(A) = k \det(B) = \det(E) \cdot \det(B);$$

(3) (Type 3) If E is obtained from I_n by adding a multiple of one row to another, then $\det(E) = 1$ and A is obtained from B by same operation. So

$$\det(A) = \det(B) = \det(E) \cdot \det(B);$$

$\square$

And now, for the main theorem about matrix multiplication and the determinant:

Theorem 15. *Suppose that $A, B \in M_{n\times n}(F)$. Then*

$$\det(AB) = \det(A) \cdot \det(B).$$

Note that this also means that $\det(AB) = \det(BA)$, although of course in general $AB \neq BA$.

Proof. **Case 1** $\det(A) = 0$. Then $rank(AB) \leq rank(A) < n$, and so $\det(AB) = 0$. The. $\det(AB) = 0 = \det(A) \cdot \det(B)$.

Case 2 $\det(A) \neq 0$. Then A is invertible. And so it is the products of elementary matrices. (We can assume these are row elementary). Write $A = E_1 ... E_k$ where each E_i is elementary. Then

$$\det(AB) = \det(E_1 \cdot ... E_k \cdot B) = \det(E_1) \det((E_2 \cdot ... E_k \cdot B) = ...$$

$$\det(E_1) \det(E_2) \cdot ... \det(E_k) \cdot \det(B) = \det(E_1 \cdot ... E_k) \det(B) = \det(A) \det(B).$$

□

Corollary 16. *If A is invertible,* $\det(A^{-1}) = \frac{1}{\det(A)}$.

Proof. Exercise. □

Finally, we note that the lemmas about elementary row operations and the determinant also hold for column operations. I.e. we have:

Fact 17. *Suppose that B is obtained from A by doing one elementary column operation. Say $B = AE$, where E is an elementary column matrix. Then, if E is of:*

(1) *Type 1, interchanging two rows:* $\det(E) = -1$, $\det(B) = -1 \det(A)$*;*
(2) *Type 2, multiplying a row by k:* $\det(E) = k$, $\det(B) = k \det(A)$*;*
(3) *Type 3:* $\det(E) = 1$, $\det(B) = \det(A)$.

In particular, if E is an elementary matrix (row or column), then

$$\det(E) = \det(E^t).$$

We leave the proof as an exercise.

Lemma 18. $\det(A^t) = \det(A)$.

Proof. If A is not invertible, then $\det(A) = 0$, and $n > rank(A) = rank(A^t)$, so A^t is not invertible and $\det(A^t) = 0$.

If A is invertible, then $A = E_1 \cdot ... \cdot E_k$, where each E_i is elementary. So

$$A^t = (E_1 \cdot ... \cdot E_k)^t = E_k^t \cdot ... \cdot E_1^t,$$

and for each $i \leq k$, $\det(E_i^t) = \det(E_i)$. Then,

$$\det(A^t) = \det(E_k^t \cdot ... \cdot E_1^t) = \det(E_k^t) \cdot ... \cdot \det(E_1^t) = \det(E_k) \cdot ... \cdot \det(E_1) = \det(A).$$

□